

CSC3060 Project 1 Report

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1 Theoretical Foundation

1.1 Binary Addition Principle

Binary addition can be decomposed into two components:

$$A + B = \text{sum without carry (Sum)} + \text{carry propagation (Carry)}$$

A	B	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Truth Table for Binary Addition

If a bit position does not generate a carry (i.e., at most one of the two bits is 1), the result can be represented by $A \oplus B$. Otherwise, if both bits are 1, a carry is generated, which is captured by $A \& B$.

The generated carry must be propagated to the next higher bit position, which is achieved by left-shifting the carry by 1. Since a carry generated in one bit position can propagate to higher bits, the algorithm iteratively processes the sum and carry until no further carries remain.

Algorithm 1: Addition via Bitwise Operations

Input: Two integers a and b

Output: The sum $a + b$

while $b \neq 0$ **do**

$\text{sum} \leftarrow a \oplus b$

$\text{carry} \leftarrow (a \& b) \ll 1$

$a \leftarrow \text{sum}$

$b \leftarrow \text{carry}$

return a

1.2 Negate

All implementations based on signed integers using two's complement representation:

$$-x = \sim x + 1$$

where $\sim x$ denotes bitwise NOT operation.

Proof

The formula can be derived from two fundamental identities:

$$\begin{cases} x + (-x) = 0 & \text{(definition of additive inverse)} \\ x + (\sim x) = -1 & \text{(bitwise complement yields all ones)} \end{cases}$$

Subtracting the second equation from the first gives:

$$[x + (-x)] - [x + (\sim x)] = 0 - (-1) \implies -x - (\sim x) = 1 \implies -x = \sim x + 1.$$

Thus, the negative of any integer can be obtained by flipping all bits and adding one.

1.3 Comparison

Since the comparison operators is not allowed in the project, their relative magnitude can be determined by analyzing the sign bit of their difference. Specifically:

$$\text{sgn}(a - b) = \begin{cases} 0 & \text{if } a \geq b \\ 1 & \text{if } a < b \end{cases}$$

where $\text{sgn}(x)$ is implemented as the bitwise operation $(x \gg 31) \& 1$.

1.4 Type Casting

The algorithm casts operands to `uint32_t` to prevent signed integer overflow (e.g., `abs(INT_MIN)`), while reusing helper functions like `add` that expect `int32_t` arguments.

This approach is valid because `static_cast` between same-width integers purely **reinterprets** the bit pattern without generating runtime instructions or altering the bits.

Furthermore, since the helper functions rely exclusively on bitwise operations, which are agnostic to the sign bit, the binary logic remains correct regardless of type interpretation.

2 Function Implementations

2.1 Addition

Let $W = 32$ denote the bit width. Subsequent analyses are implicitly based on W -bit signed integers.

As established in Section 1.1, the operation requires at most W iterations.

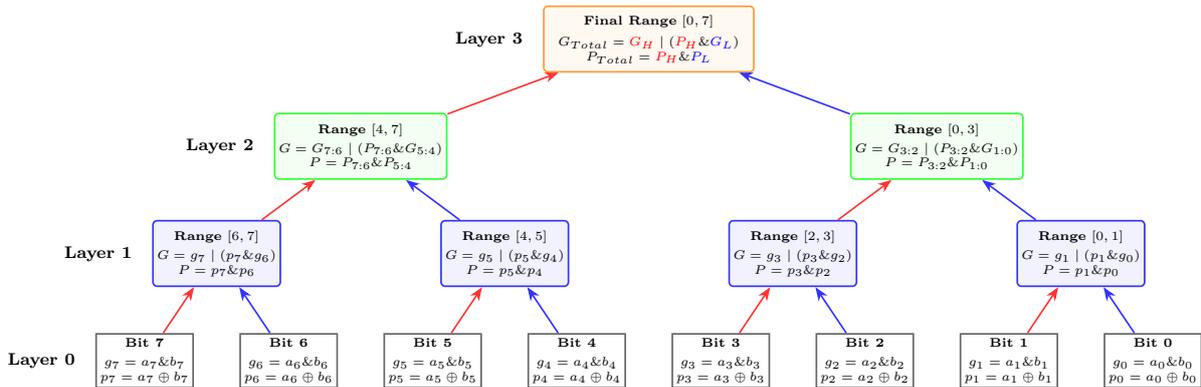
```

1 carry = (a & b) << 1;
2 a = a ^ b;
3 b = carry;

```

Thus, the addition operation can be achieved in $O(W)$ time complexity using the aforementioned operations.

However, in this algorithm, each step depends on the value generated in the previous iteration, raising the question of whether carry propagation can be computed in parallel. Modeling the computation as a tree structure allows us to apply a recursive doubling approach, effectively optimizing the carry propagation. (Similar to **Carry Lookahead Adder** mentioned in the lecture.)



An Example of 8-bit Carry Propagation

With \mathbf{G} signifying local carry creation and \mathbf{P} signifying carry transmission, the algorithm employs recursive doubling to merge carry information, achieving a time complexity of $O(\log W)$.

```

1 int32_t add (int32_t a, int32_t b)
2 {
3     int32_t p = a ^ b, g = a & b;
4     g |= (p & (g << 1)); p &= (p << 1);
5     g |= (p & (g << 2)); p &= (p << 2);
6     g |= (p & (g << 4)); p &= (p << 4);
7     g |= (p & (g << 8)); p &= (p << 8);
8     g |= (p & (g << 16));
9     return (a ^ b) ^ (g << 1);
10 }

```

2.2 Subtraction

As established in the derivation of two's complement arithmetic in Section 1.2, the subtraction $a - b$ can be transformed into $a + (-b) = a + (\sim b + 1)$.

By utilizing the already implemented `add` function, subtraction is therefore reduced to a constant number of bitwise operations, achieving $O(\log W)$ time complexity.

```
1 int32_t subtract (int32_t a, int32_t b)
2 {
3     return add (a, add (~b, 1));
4 }
```

2.3 Multiplication

The multiplication algorithm adapts traditional **column multiplication** to binary arithmetic. Analogous to shifting partial products in decimal, the binary multiplicand a is left-shifted for each active bit in the multiplier b .

Let b be represented in binary form as $b = \sum_{i=0}^{W-1} b_i \cdot 2^i$, where $b_i \in \{0, 1\}$ is the i -th bit of b . Substituting this into the product $a \times b$ yields:

$$a \times b = a \times \sum_{i=0}^{W-1} (b_i \cdot 2^i) = \sum_{i=0}^{W-1} b_i \cdot (a \cdot 2^i)$$

Before computation, signed operands are converted to their absolute values. To prevent signed integer overflow (e.g., `INT_MIN`), operands are cast to `uint32_t`.

The algorithm achieves a time complexity of $O(W \log W)$ (Both `add ()` and `subtract ()` have a time complexity of $O(\log W)$).

```
1 int32_t multiply (int32_t a, int32_t b)
2 {
3     int32_t sign = 0;
4     uint32_t tmpa = static_cast <uint32_t> (a);
5     uint32_t tmpb = static_cast <uint32_t> (b);
6     uint32_t tmpres = 0;
7     if (tmpa >> 31 & 1)
8     {
9         tmpa = subtract (0, tmpa);
10        sign ^= 1;
11    }
12    if (tmpb >> 31 & 1)
13    {
14        tmpb = subtract (0, tmpb);
15        sign ^= 1;
16    }
17    while (tmpb)
18    {
19        if (tmpb & 1) tmpres = add (tmpres, tmpa);
20        tmpa <<= 1; tmpb >>= 1;
21    }
22    int32_t res = static_cast <int32_t> (tmpres);
23    if (sign) res = subtract (0, res);
24    return res;
25 }
```

2.4 Division

Similar to the multiplication algorithm, the sign bit is processed separately by converting both operands to absolute values (using `uint32_t` is necessary).

According to **Euclidean Division Theorem**, the operation is expressed as $a = b \cdot q + r$, where $0 \leq r < b$. Let quotient q be represented in binary form as $q = \sum_{i=0}^{W-1} q_i \cdot 2^i$, where $q_i \in \{0, 1\}$ is the i -th bit of q . To determine the coefficients q_i , a **Greedy Strategy** is employed:

- **case 1** $a < b \cdot 2^i$ $b \cdot 2^i$ is too large to fit in the remainder, which implies $q_i = 0$.
- **case 2** $a \geq b \cdot 2^i$ It implies $q_i = 1$. The i -th bit of the result is set, and the remainder is updated to $a \leftarrow a - b \cdot 2^i$.

Proof

Suppose that q_i is set to 0 despite the condition $a \geq b \cdot 2^i$.

The value of r would satisfy the inequality:

$$r = a - \sum_{j=0}^{i-1} q_j \cdot b \cdot 2^j \geq a - b(2^i - 1)$$

Substituting the condition $a \geq b \cdot 2^i$ into the inequality:

$$r \geq b \cdot 2^i - b(2^i - 1) = b$$

This leads to $r \geq b$, which contradicts the Euclidean division constraint $0 \leq r < b$.

The comparison outlined in Section 1.3 should be utilized to avoid invalid behaviors. And the algorithm achieves a time complexity of $O(W \log W)$.

```
1 int32_t divide (int32_t a, int32_t b)
2 {
3     int32_t sign = 0;
4     uint32_t tmpa = static_cast <uint32_t> (a);
5     uint32_t tmpb = static_cast <uint32_t> (b);
6     uint32_t tmpres = 0;
7     if (tmpa >> 31 & 1)
8     {
9         tmpa = subtract (0, tmpa);
10        sign ^= 1;
11    }
12    if (tmpb >> 31 & 1)
13    {
14        tmpb = subtract (0, tmpb);
15        sign ^= 1;
16    }
17    for (int i = 31; ~i; --i)
18    {
19        int32_t diff = subtract (tmpa >> i, tmpb);
20        if (!(diff >> 31 & 1))
21        {
22            tmpres |= (1 << i);
23            tmpa = subtract (tmpa, tmpb << i);
24        }
25    }
26    int32_t res = static_cast <int32_t> (tmpres);
27    if (sign) res = subtract (0, res);
28    return res;
29 }
```

2.5 Modulo

In mathematical number theory, the modulo operation is defined as $a \bmod b = a - b \cdot \lfloor \frac{a}{b} \rfloor$. But in C++, the modulo operation follows the **truncate toward zero** rule. Note that the `divide (a,b)` is also implemented to align with this C++ standard (truncate toward zero rule), the Euclidean relation $a = qb + r$ can be reversed to derive the correct modulus $r = a - qb$. The time complexity of this algorithm is $O(W \log W)$.

```
1 int32_t modulo (int32_t a, int32_t b)
2 {
3     int32_t q = divide (a, b);
4     int32_t mx = multiply (b, q);
5     return subtract (a, mx);
6 }
```

3 Unit Test

Following initial verification, several test cases were rigorously enhanced with LLM (prompts in Appendix) to focus on three critical dimensions:

3.1 Boundary Value Analysis

- **Extremes:** Operations involving `INT_MAX` and `INT_MIN`.
- **Zero Handling:** Rigorous testing of 0 as both an operand and a result

3.2 Randomized Mixed-Sign Testing

- **Full-Range Inputs:** Random integers selected from the entire `int32_t` range.
- **Mixed Signs:** Combinations of positive and negative to ensure correctness.

3.3 Truncation and Sign Consistency

The tests rigorously verify the sign correctness for all four permutations of input signs: $a \bmod b$, $a \bmod (-b)$, $(-a) \bmod b$ and $(-a) \bmod (-b)$.

3.4 Flow Test

A dedicated suite of flow tests was designed to verify the implementation's behavior under overflow and underflow conditions.

4 Appendix

It is hereby declared that Generative AI tools were utilized in this project to assist with code optimization, report polishing, and test case generation. The core logic and implementation were verified by the author. **Representative excerpts** of the interaction are provided below:

Report Translation and Polishing

If a bit position does not generate a carry (i.e., at most one of the two bits is 1), the result can be represented by $a \oplus b$. Otherwise, if both bits are 1, a carry is generated, which is captured by $a \& b$.
Once get a carry, it can be used left shift to put it in higher bit. The expression is :
润色这句话 给我 markdown 源码 ""md ""

```
markdown
### Binary Addition Logic
1. No Carry Case:
   If a bit position does not generate a carry (i.e., when at most one of the two bits is '1'), the result is simply the XOR of the two bits.
   \[
   \text{result} = a \oplus b
   \]
2. Carry Generation:
   If both bits are '1', a carry is generated. This carry is captured by the bitwise AND operation.
   \[
   \text{carry} = a \& b
   \]
3. Carry Propagation:
   The generated carry must be propagated to the next higher bit position. This is achieved by left-shifting the carry.
   \[
   \text{carry\_propagated} = (a \& b) \ll 1
   \]
### Final Expression
The complete binary addition can be expressed as:
\[\text{result} = (a \oplus b) + ((a \& b) \ll 1)\]
```

在通过所有基础测试后,借助 AI (prompt 会放入附录中) 增加了更强的数据,包括:
1. 边界值与特殊情况 包括 INT_MAX, INT_MIN, 0 的加入
2. 增加正负数综合的随机数据
3. 对于取模, 需要保证 truncation toward zero 的正确性, 对 $a \% b$, $(-a) \% b$, $(-a) \% (-b)$ 分别进行测试
写入我的报告中 并润色 不要出现 I we 等词语, 放入 \section{Unit Test} 中

Here is the LaTeX source for the **Unit Test** section, incorporating your specific points about AI-assisted enhancement and rigorous coverage. I have polished the language to be professional and passive-voice, as requested.

```
latex
\section{Unit Testing Strategy}
To ensure the correctness and robustness of the bitwise arithmetic implementation, a comprehensive testing strategy is employed, covering various edge cases and random inputs.
\subsection{Boundary Value Analysis}
Due to the inherent asymmetry of the Two's Complement representation, arithmetic errors most frequently occur at the boundaries of the integer range. Rigorous testing is performed for:
\begin{itemize}
\item Extremes: Operations involving  $\text{INT\_MAX}$  and  $\text{INT\_MIN}$ .
\item Zero Handling: Rigorous testing of 0 as both an operand and a result (e.g.,  $0 \oplus a$ ,  $0 \& a$ ,  $0 \ll 1$ ).
\item Overflow Conditions: Specific scenarios such as  $\text{INT\_MIN} / -1$  and  $\text{INT\_MAX} + 1$ .
\end{itemize}
\subsection{Randomized Mixed-Sign Testing}
To validate behavior across the entire 32-bit integer spectrum, a randomized testing engine was used to generate a wide variety of inputs, including:
\begin{itemize}
\item Full-Range Inputs: Random integers selected from the entire  $\text{int32\_t}$  range.
\item Mixed Signs: Combinations of (positive, positive), (positive, negative), (negative, positive), and (negative, negative).
\end{itemize}
The results of the custom functions are cross-verified against the standard C++ arithmetic operators.
\subsection{Truncation and Sign Consistency}
Particular attention was given to the division and modulo operations to ensure compliance with the C++ standard's truncation toward zero.
\begin{itemize}
\item  $a \% b$ 
\item  $a \% (-b)$ 
\item  $(-a) \% b$ 
\item  $(-a) \% (-b)$ 
\end{itemize}
This ensures that the relation  $a \% b = (a/b) \cdot b + (a \% b)$  holds true for all inputs, confirming the correctness of the implementation.
```

Add Content

`\subsection{Type Casting}`
The algorithm relies on casting operands to `\texttt{uint32_t}` to prevent signed overflow (e.g., `\texttt{abs(INT_MIN)}`), while utilizing helper functions like `\texttt{add}` that expect `\texttt{int32_t}` arguments.

请帮我继续完善并简洁的阐述, 说明这样子是正确的, 因为 `static_cast` 的本质是做规则检查, 不会产生运行时的指令, 而所有函数只有位运算操作, 因此不会出现截断等行为

Here is a concise and complete version for your report. It clearly explains the correctness based on bit preservation and compiler behavior.

```
latex
\subsection{Type Casting Strategy}
The algorithm casts operands to \texttt{uint32\_t} to prevent signed integer overflow (e.g., \texttt{abs(INT\_MIN)}). This approach is valid because \texttt{static\_cast} between same-width integers purely \texttt{reinterprets} the bits, without generating any runtime instructions or altering the underlying bit patterns.
```

Key Points Included:

- Reinterpretation**: Emphasized that `static_cast` only changes how bits are viewed, not the bits themselves.
- No Overhead**: Mentioned that it doesn't generate extra runtime instructions.
- Sign Agnosticism**: Explained that bitwise operators don't care about types, ensuring the logic holds.

Test Case Enhancement

```

}

TEST(ModuloTest, RandomNumber2) {
    EXPECT_EQ(data_lab::modulo(-1205291292, -6441909970), -561100322);
}

在通过所有基础测试后, 包括:
1. 边界值与特殊情况 包括 INT_MAX, INT_MIN, 0 的加入
2. 增加正负数混合的随机数据
3. 对于取模, 需要保证 truncation toward zero 的正确性, 对 a % b, a % (-b), (-a) % b, (-a) % (-b) 分别进行测试

直接创建一个类用于生成各种随机数据/检验数据合法性/边界情况

然后基于 TEST 生成 TEST_F 五类数据直接形成子类不需要压力测试 也不需要不合理的溢出
    
```

```

cpp 位置

#include <gtest/gtest.h>
#include <stdint>
#include <random>
#include <vector>
#include <algorithm>
#include <limits>
#include <chrono>
#include "../include/data_lab.hpp"

class BaseTest : public ::testing::Test {
protected:
    void SetUp() override {
        gen = std::mt19937(static_cast<unsigned int>(std::chrono::high_resolution_clock::now().time()));
    }
    int32_t rand_full() { return std::uniform_int_distribution<int32_t>(INT32_MIN, INT32_MAX)(gen); }
    int32_t rand_safe() { return std::uniform_int_distribution<int32_t>(INT32_MIN/2, INT32_MAX/2)(gen); }
    int32_t rand_pos() { return std::uniform_int_distribution<int32_t>(1, INT32_MAX)(gen); }
    int32_t rand_neg() { return std::uniform_int_distribution<int32_t>(INT32_MIN, -1)(gen); }
    int32_t rand_nonzero() {
        int32_t v; do { v = rand_full(); } while(v == 0); return v;
    }
    int32_t rand_small() { return std::uniform_int_distribution<int32_t>(-1000, 1000)(gen); }
    bool add_overflow(int32_t a, int32_t b) { return (b > 0 && a > INT32_MAX - b) || (b < 0 && a < INT32_MIN + b); }
    bool sub_overflow(int32_t a, int32_t b) { return (b > 0 && a < INT32_MIN + b) || (b < 0 && a > INT32_MAX - b); }
    bool mul_overflow(int32_t a, int32_t b) { int64_t r = (int64_t)a * b; return r < INT32_MIN || r > INT32_MAX; }
};

class AddTest : public BaseTest {};
class SubtractTest : public BaseTest {};
class MultiplyTest : public BaseTest {};
class DivideTest : public BaseTest {};
class ModuloTest : public BaseTest {};

TEST_F(AddTest, Basic) {
    EXPECT_EQ(data_lab::add(5, 3), 8);
    EXPECT_EQ(data_lab::add(-5, -3), -8);
    EXPECT_EQ(data_lab::add(10, -3), 7);
    EXPECT_EQ(data_lab::add(0, 5), 5);
    EXPECT_EQ(data_lab::add(0, 0), 0);
    EXPECT_EQ(data_lab::add(0x55555555, 0xA4444444), -1);
    EXPECT_EQ(data_lab::add(0xFFFFFFFF, 1), 0);
}

TEST_F(AddTest, boundaries) {
    EXPECT_EQ(data_lab::add(INT32_MAX - 1, INT32_MAX);
    EXPECT_EQ(data_lab::add(INT32_MIN + 1, -1), INT32_MIN);
}

TEST_F(AddTest, Random) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_safe(), b = rand_safe();
        if (add_overflow(a, b)) EXPECT_EQ(data_lab::add(a, b), a + b);
    }
}

TEST_F(AddTest, Properties) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_full(), b = rand_full();
        EXPECT_EQ(data_lab::add(a, b), data_lab::add(b, a));
        int32_t x = rand_safe()/3, y = rand_safe()/3, z = rand_safe()/3;
        EXPECT_EQ(data_lab::add(data_lab::add(x, y), z), data_lab::add(x, data_lab::add(y, z)));
    }
}
    
```

```

}

TEST_F(MultiplyTest, Basic) {
    EXPECT_EQ(data_lab::multiply(5, 3), 15);
    EXPECT_EQ(data_lab::multiply(-5, -3), 15);
    EXPECT_EQ(data_lab::multiply(5, -3), -15);
    EXPECT_EQ(data_lab::multiply(0, -5), 0);
    EXPECT_EQ(data_lab::multiply(7, 1), 7);
    EXPECT_EQ(data_lab::multiply(7, -1), -7);
}

TEST_F(MultiplyTest, Boundaries) {
    EXPECT_EQ(data_lab::multiply(INT32_MAX, 1), INT32_MAX);
    EXPECT_EQ(data_lab::multiply(INT32_MIN, 1), INT32_MIN);
}

TEST_F(MultiplyTest, Random) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_small(), b = rand_small();
        if (mul_overflow(a, b)) EXPECT_EQ(data_lab::multiply(a, b), a * b);
    }
}

TEST_F(MultiplyTest, Properties) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_small(), b = rand_small(), c = rand_small();
        EXPECT_EQ(data_lab::multiply(a, b), data_lab::multiply(b, a));
        EXPECT_EQ(data_lab::multiply(a, data_lab::add(b, c)),
            data_lab::add(data_lab::multiply(a, b), data_lab::multiply(a, c)));
    }
}

TEST_F(DivideTest, Basic) {
    EXPECT_EQ(data_lab::divide(15, 3), 5);
    EXPECT_EQ(data_lab::divide(-15, -3), 5);
    EXPECT_EQ(data_lab::divide(-15, 3), -5);
    EXPECT_EQ(data_lab::divide(0, 5), 0);
    EXPECT_EQ(data_lab::divide(42, 1), 42);
    EXPECT_EQ(data_lab::divide(INT32_MIN, 1), INT32_MIN);
    EXPECT_EQ(data_lab::divide(-7, 2), -3);
    EXPECT_EQ(data_lab::divide(7, -2), -3);
    EXPECT_EQ(data_lab::divide(1, 2), 0);
}

TEST_F(DivideTest, Random) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_full(), b = rand_nonzero();
        if (a == INT32_MIN && b == -1) continue;
        EXPECT_EQ(data_lab::divide(a, b), a / b);
    }
}

TEST_F(DivideTest, Relation) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_full(), b = rand_nonzero();
        if (a == INT32_MIN && b == -1) continue;
        int32_t a = data_lab::divide(a, b), r = data_lab::modulo(a, b);
        EXPECT_EQ(data_lab::add(data_lab::multiply(a, b), r), a);
    }
}

TEST_F(ModuloTest, Basic) {
    EXPECT_EQ(data_lab::modulo(17, 5), 2);
    EXPECT_EQ(data_lab::modulo(-17, 5), -2);
    EXPECT_EQ(data_lab::modulo(17, -5), 2);
    EXPECT_EQ(data_lab::modulo(-17, -5), -2);
    EXPECT_EQ(data_lab::modulo(0, 5), 0);
    EXPECT_EQ(data_lab::modulo(7, 3), 1);
    EXPECT_EQ(data_lab::modulo(-7, 3), -1);
}

TEST_F(ModuloTest, Random) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_full(), b = rand_nonzero();
        if (a == INT32_MIN && b == -1) continue;
        EXPECT_EQ(data_lab::modulo(a, b), a % b);
    }
}

TEST_F(ModuloTest, RangeCheck) {
    for (int i = 0; i < 100; ++i) {
        int32_t a = rand_full(), b = rand_nonzero();
        if (a == INT32_MIN && b == -1) continue;
        int32_t r = data_lab::modulo(a, b);
        EXPECT_TRUE(abs(r) < abs(b) || r == 0);
        if (r != 0) EXPECT_TRUE((a > 0) == (r > 0));
    }
}
    
```

Auxiliary Image Generation

This diagram is the final result of implementing the illustration in Section 2.1, synthesized after multiple rounds of prompt refinement.

这个算法是并行计算结果，类似借位进行答案的合并

也有问题吧 应该画的像树状数组/线段树这种结构才对1吧
我觉得那个配图仍然没有很好的说明这段代码的本质 能否再细化一下
(以三层的为例子)
应该画一个圈 底下有 8 个节点表示每一个 bit
然后往上一层节点数 / 2
每一层跨度也再变大 所以是 1 2 4 8
直接在树里面体现进位的情况

去除 logic 说明，加一个标题 a example of 8 bit

统一改为 layer 0, layer1 ,...

标题放在底部

去除代码注释

减少每一层箭头长度

稍微一句话阐述一下算法 (最后说明复杂度) 首先要定义 g,p 的含义 英文

推理完成 >

好的，已根据你的要求进行了调整：

1. 层级命名：统一修改为 **Layer 0, Layer 1, Layer 2, Layer 3** (保留了括号内的 k 值以便理解步长)。
2. 标题位置：移动到了图的**底部**。
3. 箭头长度：将全局 y 比例从 1.8cm 压缩到 1.2cm，并将层间偏移量从 2 减小到 1.5，使图形更紧凑。
4. 清理代码：去除了所有解释性的代码注释。

请完全覆盖你的 tree.tex：

```
latex
\begin{tikzpicture}
  \x=1.5cm, \y=1.2cm,
  node distance=1.5cm,
  leaf/.style={
    rectangle,
    draw=black!50,
    thick,
    fill=white,
    minimum width=1.35cm,
    minimum height=0.9cm,
    font=\bfseries\tiny,
    align=center,
    inner sep=2pt
  },
  mergebox/.style={
    rectangle,
    draw=blue!80,
    fill=blue!5,
    thick,
    rounded corners=2pt,
    minimum width=1.8cm,
    minimum height=1.1cm,
    align=center,
    font=\bfseries\tiny,
    inner sep=2pt
  },
  arrow/.style={
    ->,
    >stealth,
    thick,
    color=gray!70
  }
}

\foreach \i in {7,6,...,0} {
  \node[leaf] (b\i) at {(7-\i)*1.6, 0} {
    Bit \i \\ [2pt]
    $G_{(\i)} = a_{(\i)} \wedge b_{(\i)}$ \\
    $P_{(\i)} = a_{(\i)} \vee b_{(\i)}$
  };
}

\node[left=0.2cm of b7, font=\bfseries\scriptsize, align=right] (Layer 0);

\foreach \i in {7,5,3,1} {
  \gfmthparse(int(\i-1))
  \let\j\gfmthresult

  \node[mergebox] (L\i) at {$(b\i)\wedge(b\j) + (0, 1.5)$} {
    \textbf{Range} $[0, \i]$ \\ [2pt]
    $G = G_{(\i)} \wedge P_{(\j)} \wedge G_{(\i+1)}$ \\
    $P = P_{(\i)} \wedge P_{(\j)}$
  };

  \draw[arrow, red!80] (b\i) -- (L\i);
  \draw[arrow, blue!80] (b\j) -- (L\i);
}

\node[left=0.2cm of L1_7, font=\bfseries\scriptsize, align=right] (Layer 1 \i\{k=1\});

\node[mergebox, fill=green!5, draw=green!80] (L2_7) at {$(L1_7)\wedge(L1_5) + (0, 1.5)$} {
  \textbf{Range} $[4, 7]$ \\ [2pt]
  $G = G_{(7:6)} \wedge P_{(7:6)} \wedge G_{(5:4)}$ \\
  $P = P_{(7:6)} \wedge P_{(5:4)}$
};

\draw[arrow, red!80] (L1_7) -- (L2_7);
\draw[arrow, blue!80] (L1_5) -- (L2_7);

\node[mergebox, fill=green!5, draw=green!80] (L2_3) at {$(L1_3)\wedge(L1_1) + (0, 1.5)$} {
  \textbf{Range} $[0, 3]$ \\ [2pt]
  $G = G_{(3:2)} \wedge P_{(3:2)} \wedge G_{(1:0)}$ \\
  $P = P_{(3:2)} \wedge P_{(1:0)}$
};

\draw[arrow, red!80] (L1_3) -- (L2_3);
\draw[arrow, blue!80] (L1_1) -- (L2_3);

\node[left=0.2cm of L2_7, font=\bfseries\scriptsize, align=right] (Layer 2 \i\{k=2\});

\node[mergebox, fill=orange!5, draw=orange!80, minimum width=3.5cm] (L3_7) at {$(L2_7)\wedge(L2_3)$} {
  \textbf{Final Range} $[0, 7]$ \\ [2pt]
  $G_{\text{Total}} = \text{color}(red)\{G_{(0)}\} \wedge \text{color}(red)\{P_{(4)}\} \wedge \text{color}(blue)\{G_{(1)}\}$ \\
  $P_{\text{Total}} = \text{color}(red)\{P_{(0)}\} \wedge \text{color}(blue)\{P_{(1)}\}$
};

\draw[arrow, red!80] (L2_7) -- (L3_7);
\draw[arrow, blue!80] (L2_3) -- (L3_7);

\node[left=0.2cm of L3_7, font=\bfseries\scriptsize, align=right] (Layer 3 \i\{k=3\});

\node[font=\bfseries\large, align=center] at (5.5, -1.2) {An Example of 8-bit Carry Propagation};

\end{tikzpicture}
```

Problem Assistance

为什么我直接用 $a - (a / b) * b$ 实现 $a \% b$ 无法通过测试数据
C++ 取余不是负数的时候会有特殊的问题啊

推理完成 >

第一部分：为什么 $a - (a / b) * b$ 会挂？

公式 $r = a - (a/b) \times b$ 在数学逻辑上是完全正确的，它是 C++ 标准中取余 ($\%$) 的定义。

如果你的代码无法通过测试，通常是因为以下 3 个原因之一：

1. 除法 (divide) 的实现行为不符合 C++ 标准
 - C++ 标准：整数除法是向 0 取整 (Truncate toward zero)。
 - 例如： $-7 / 2$ 应该等于 -3 (而不是 -4)。
 - 问题所在：如果你的 divide 函数在处理负数时使用了向下取整 (Floor) (例如通过位移 \gg 实现且没处理符号位，或者模仿了 Python 的除法)，那么算出来的 q 就是错的，导致算出来的余数 r 也是错的。
 - 检查方法：检查 $\text{divide}(-7, 2)$ 是否返回 -3 。
2. 中间结果溢出
 - 虽然最终的余数肯定能存下，但中间步骤 $q * b$ 可能会出现溢出问题，或者 $a - (q*b)$ 在减法实现中处理溢出有问题。
 - 特殊情况： `INT_MIN % -1`。
 - 在 C++ 中，`INT_MIN / -1` 会导致浮点异常 (FPE) 或溢出 (因为结果是 `INT_MAX + 1`)。
 - 如果在你的 divide 实现中没有特判这种情况，程序可能会崩溃或算出错误的值。
3. 符号位的处理
 - 在 C++ 中，余数的符号必须与被除数 (a) 的符号一致。
 - $7 \% 3 = 1$
 - $-7 \% 3 = -1$
 - $7 \% -3 = 1$
 - $-7 \% -3 = -1$
 - 如果你的 divide 也是基于先取绝对值算，最后加符号的逻辑，那么 $a - (a/b)*b$ 这个公式是天然正确的，不需要额外处理符号。但如果 divide 实现有偏差，余数符号就会反。

Other Links

Link1 Link2 link3